Eleven-year-old Paul reached into a paper bag that he knew contained two pink plastic cubes and four blue plastic cubes. Without looking, he stirred the contents before drawing out a pink cube, a result that contradicted the “blue” draw he had predicted. Three previous trials with other color combinations had also failed to support his predictions. When I asked him to explain these results, Paul said, “It’s really strange. Everything is playing tricks on us!”

Paul is learning to deal with probability, a field of mathematics that, along with its close relation, statistics, influences the decisions we make both as individuals and as a society (Paulos 1988). Recognizing the importance of this branch of mathematics, many countries around the world now include probability theory in the elementary school curriculum (Amir and Williams 1999). NCTM’s Principles and Standards for School Mathematics (2000, p. 181) recommends that probability instruction be introduced informally in kindergarten and states that students in grades 3 to 5 “can begin to learn how to quantify likelihood.” According to some researchers, however, the optimal timing and content of probability instruction in elementary and middle school is “still an open

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As an elementary teacher and researcher, I am fascinated by children’s intuitive responses to probability. Many fourth and fifth graders—with Santa Claus and the Tooth Fairy still fresh in their minds—maintain a magical outlook on life. How well do they understand randomness and chance, theoretical and experimental probability, and independent and conditional events?

As part of an investigation into children’s probabilistic thinking, I spent three, forty-minute after-school sessions with Paul, one of my fifth-grade students, videotaping his responses to a series of activities using cubes, coins, dice, and spinners. The sessions, which took place in our classroom, were part of my university graduate work in mathematics education and provided a pilot project for a master’s thesis case study with my complete fourth- and fifth-grade class. Paul, like most of the approximately three hundred students in our semi-rural school, is from a white, middle-class, English-speaking background. He had explored probability briefly in fourth grade but had not encountered probability in fifth grade at the time of the interviews.

Like many elementary students, Paul is clear about some aspects of probability and confused about others. His responses demonstrate a mix of sophisticated insights and surprising misconceptions.

### Tossing Coins

After probing Paul’s general ideas about chance, I introduced questions about flipping a penny. Many commonly recommended activities for teaching probability are based on the assumption that children perceive heads and tails to be equally likely outcomes of a coin toss. Paul’s comments show that this assumption is not always valid.

**Teacher.** If I flip this penny, what are the possible things that could happen? What are the possible outcomes?

**Paul.** Well, it could land on heads, it could land on tails, or—it would be very improbable that it could land on its side like this. *(He demonstrates by holding the penny on its edge.)* I’ve only had that happen to me once before.

**Teacher.** You’ve actually had that happen?

**Paul.** Yeah, but it was with a bit of a thick coin.

**Teacher.** Mmm, it would have to be, I think. So, what are the chances that we’re going to toss heads?

**Paul.** Well, of all the times I’ve tossed a coin, mostly it’s been heads.

I accepted without comment Paul’s statement that he usually tosses heads and asked him to conduct a trial of ten tosses, which resulted in six tails and four heads. When asked to explain this result, Paul launched into a complicated but precise explanation.

**Paul.** I’m not sure, but I think the reason they land on a certain side is because of the weight of one side. When it’s flipping, one side becomes heavier so it drops on that side and, if you look, heads has more on it. So it’s heavier on that side, so when it comes down like this [on the table] it’s tails and when we put it in our hand and flip it over *(flips coin from right palm to back of left hand)* it’s heads—because it turns to the opposite side.

Clearly, Paul’s comments show that he is used to seeking and finding rational explanations for real-world events. His thoughts about coin tossing are based on deterministic, rather than probabilistic, thinking. They also reveal a creative ability to interpret events to fit preconceived notions. In this case, Paul managed to turn tails into heads (with the classic hand flip) in order to support his self-professed tendency to toss heads!

Next, I wanted to check Paul’s thoughts about the connection between theoretical probability and the results of a particular experiment. In order to work successfully with probability, children must recognize that a single trial of a random process such as coin tossing is essentially unpredictable. At the same time, they must keep in mind that the overall outcome of many trials can be predicted using probability theory. This concept of “predicting the unpredictable” is a major challenge for anyone studying probability.

**Teacher.** If we did one hundred tosses, what would you expect to have happen? How many heads and tails?

**Paul.** About sixty tails, because with this test we got six out of ten. But to get a really accurate estimation you would have to do about two or three 10 ones [trials of ten tosses] to be really accurate. But if you just do it once, it could be anything, right? But if it keeps going in one way then you know.

Coins, dice, and spinners are used to create ran-
dom, equally likely outcomes for experiments in classical probability. In contrast, experiments such as recording how many times a dropped paper cup lands on its side deal with observed frequency. Paul’s comments reveal that he recognizes that greater numbers of trials provide the basis for more accurate predictions. Instead of focusing on the theoretically equal likelihood of tossing tails, however, Paul based his prediction on the observed results of the first trial.

Before putting the penny away, in one last bid to check for the concept of equally likely outcomes, I asked Paul why coin tosses are used to make decisions. The philosophical slant of his final answer made me stop and think.

Teacher. If somebody says, “OK, we’ve got one piece of candy here and I’ll flip you for it,” what do they mean?

Paul. They mean they’ll flip a coin and you’ll call it. And if that’s not it then they get the candy, right? It’s kind of like a game of chance to see who gets the candy.

Teacher. Is it a fair game of chance?

Paul. Well, I never think that chance is fair because in chance anything can happen.

**Drawing from Candy Bags**

In our second session I asked Paul to choose between pairs of “candy bags” containing different combinations of blue and pink plastic cubes, representing blueberry and raspberry candies. He was to pretend that he preferred raspberry and make a choice based on which bag would give him the best chance of getting that flavor. The first choice was between bag A, containing one pink cube and four blue cubes, and bag B, containing two pink cubes and three blue cubes (see fig. 1).

Teacher. If you had to reach in without looking, and just by chance pull out a candy, which bag would you choose?

Paul. This one, obviously (points to B), because if you reach in this one [A], you have a high chance of getting blueberry because there are more of them in there. And this one [B]—there are more raspberries so you have a higher chance of getting raspberry because there are more of them in there.

Paul immediately recognized that he had a better chance of pulling a pink cube from bag B (although his reason, “because there are more of them in there,” is stated in terms of absolute amounts rather than ratios). When he pulled one cube from each bag, however, the result—pink from bag A and blue from bag B—was the exact opposite of what he expected.

Teacher. So what happened? You knew this bag [B] gave you a better chance of getting pink than this bag [A]. How would you explain what actually happened?

Paul. Well, I don’t know (laughs). I’m really confused about that because you have a higher chance of getting it in there [bag B]. But like I said, it’s all chances—you really can’t say what’s going to happen.

In this response, Paul began to tackle the challenge of reconciling unpredictability in the short term (with just a few events) with predictability in the long term (over many events). His final com-
ment, “You really can’t say what’s going to happen,” is true for this situation with just one trial per bag. Paul became confused, however, when he tried to merge the two ideas that although you can say that there is a greater chance of pulling blue from bag A, you cannot make an exact prediction for just one draw.

After presenting Paul with other pairs of “candy bags” containing pink and blue cubes in various ratios, I again attempted to check his understanding of the relationship between experimental and theoretical probability.

Teacher. What do you think would happen if, instead of having ten candies in this bag, we had one hundred: fifty pink and fifty blue? What would you expect if we pulled out sixty candies? How many pinks would you expect?

Paul. Probably about twenty. Because it seems I’m really getting blues even though I’m supposed to be looking for raspberries—uh, pinks.

As he had done with the coin tosses, Paul based his prediction on the results of earlier trials. His comment of “I’m really getting blues” is reminiscent of the common misconception that past trials will influence the outcome of future events.

**Rolling Dice**

Later in the second session, we switched from candy bags to dice. Paul’s responses to my questions about rolling one die again showed deterministic thinking tinged with superstition.

Teacher. Now I want to ask you, when we roll one ordinary die, what are the possible things we can get? What are the possible outcomes?

Paul. Well, we can get from one to six. Usually I get a four or a three.

Teacher. Why is that, do you think?

Paul. I’m not sure. They’re right in the middle.

Teacher. What’s the hardest number to get—or is there a hardest number?

Paul. Well, it’s supposed to be a six but the hardest to get is a one.

Knowing Paul’s interest in games, I believe the idea that a one is difficult to roll could stem from playing games in which rolling a one is particularly desirable. Compared to all five other possible outcomes combined, rolling a one is unlikely. Also, in games with two dice, double ones (or any doubles) have a low probability of occurring. In fact, the chance of rolling “snake eyes” is only one in thirty-six. Intuitively, this could lead to the idea that, because double ones are difficult to roll, single ones must also be difficult. Paul may also be aware that the lowest and highest numbers (two and twelve) are the least likely outcomes when rolling two dice, and he may be transferring that knowledge to the single-die situation. As Fischbein (1975) points out, our intuitive ideas about probability and other mathematical concepts are based on everyday experience and often mislead us during formal learning.

**Spinning Spinners**

Our third and final session focused on spinners, another common tool for exploring classical probability. For each spinner, we used a circle divided into six equal parts and a paper clip twirled around the point of a pencil. The first spinner, A (see fig. 2), had one segment shaded gray. Paul was easily able to state that one-sixth of the spinner was shaded and predicted that ten of sixty spins would land on the shaded part. Asked if it would be possible to have all six spins come up gray, Paul answered, “Yes, but I think it’s very, very unlikely.” Hearing these answers, I began to think that Paul’s probabilistic thinking had suddenly matured.

Paul’s responses to spinner B (see fig. 2), with one-half shaded, made me realize, however, that misleading intuitions were still playing a role in his thinking. Asked to choose white or gray, Paul said, “To tell the truth, they’re both exactly the same, so it’s really hard to choose but I would probably say the shaded part.” Why? “I don’t know, but I’ve used a magnet and the loose pencil lead was pulled up by it.” Here, he suspiciously inspected the paper clip. The next bit of dialogue shows that Paul’s predictions were still influenced by previous results.

Teacher. If you had one hundred spins, what would you expect?

Paul. I have no idea!

Teacher. If you knew you were going to do tons and tons of spins, which color would you pick: white or gray?

Paul. I’m not sure. If I was doing it for something important I would probably have to think really hard on it.
Teacher. And what do you think you would decide?

Paul. I seem to get white more often. I seem to get white no matter what board I use, and no matter how many grays there are. So I’d probably use white.

I then introduced spinner C (see fig. 2), which had alternating segments shaded gray. Paul commented that it looked like “the radioactive sign.” Although he recognized that this new spinner was also half, or three-sixths, shaded, he did not conclude that it would give the same results as spinner B. Asked to compare spinners B and C, he said, “I’m not sure, but I think this one [C] would probably get more grays than this one [B] because it [C] is all over the place so you have a higher chance of a number. Otherwise, on this one [B], when you spin it around it either goes on one or the other.”

Other researchers (Green 1983; Jones et al. 1997) have noted the misconception that spinners with noncontiguous segments produce different probabilities than spinners with contiguous segments of the same area. Instead of seeing a spinner as a model of randomness, children like Paul appear to focus on visual and physical clues, trying to relate them to their own experience. It is almost as though they are pondering the travels of the spinner needle as they would their own movements. Which is more likely to slow you down: one big puddle or two little ones?

Our last experiment used spinner D (see fig. 2), which was one-sixth striped, two-sixths spotted, and three-sixths plain white. Like spinner C, this spinner alternated patterned and white segments. Again, Paul’s answers show a mix of mature and immature thinking about probability. Although I had done no direct teaching during the three sessions and had tried to avoid leading comments and questions, his responses to this final experiment seem to show increased awareness of randomness and theoretical probability. The depth of Paul’s understanding could not be determined, however, from the videotaped sessions.

Teacher. If you looked at this spinner [D] and you had to pick either spots, plain, or stripes to be your pattern and if you landed on it you would get a point, which would you pick?

Paul. Spots.

Teacher. Can you tell me why?

Paul. I don’t know because it’s sort of evenish.

Teacher. So if you’re spinning you would have more chance of landing on spots?

Paul. Well, possibly. The logical answer would be white.

Teacher. Why?

Paul. Because there’s more white than anything else.

Conclusion

Two months after conducting these interviews with Paul, I carried out a three-week instructional unit on probability with my whole class of twenty-seven fourth- and fifth-grade students. In six, fifty-minute lessons, the children used dice and spinners to investigate probability through games and experiments. We used commonly recommended
activities for this age level such as a two-dice “horse race” game (Steinbring 1991) and a “spinner sums” activity (Burns 1995). One session also focused on comparing two spinners with contiguous and noncontiguous segments (similar to those in fig. 3).

Throughout the unit, students made predictions, recorded actual results, pooled their results to produce larger samples, discussed their findings, and wrote about their experiences in their mathematics journals. I videotaped the sessions and collected copies of student work as part of a case study for my master’s thesis. Analyzing student responses before, during, and after this instructional unit, I found that, although some students were able to apply probabilistic thinking in some contexts, many showed the same subjective, causal, and deterministic ideas that Paul displayed during these earlier interviews.

Unlike many basic mathematical concepts, probability is difficult to learn through direct modeling or trial and error (Borovcnik and Peard 1996). Interpreting the results of our games and experiments often was not easy. Was an unbalanced outcome due to randomness and chance, or the way some children deliberately rolled the dice? Or could it be that our dice were actually “loaded”?

This project and others (Fischbein and Gazit 1984; Watson and Moritz 2003) show that concrete probability experiments may confirm a child’s faulty intuitions or even undermine concepts that he or she has already begun to acquire. On a posttest question asking children to advise “Mia” about two equivalent spinners (see fig. 3), for example, Paul was one of fourteen children who recommended one spinner over the other. He answered, “From past results I can tell her spinner B will give her best results.” Four other students also chose spinner B and nine picked spinner A, with explanations such as “I think from our studies that you should choose spinner A because the chances of getting striped are more bunched up.”

In light of these findings and the pressures of an over-packed curriculum, I wonder about the advisability of spending classroom time on direct investigations of abstract concepts of classical probability. I believe that my students need more time to develop the concepts that underlie statistics and probability: a strong sense of whole numbers and fractions, multiplicative thinking, estimation skills, ideas of ratio and proportion, and intuitions about random distributions in everyday life.

Instead of relying on dice and spinners, I plan to follow the advice of researchers who recommend that probability discussions be embedded in data analysis related to real-world investigations (Ahlgren and Garfield 1991; Shaughnessy 2003). Recently, for example, my class created a typical pictograph showing the students’ birthdays, sorted by month. The graph provided the data for a probability activity. We put everyone’s names in a paper bag and I asked the children to use both words and numbers to describe the probability of drawing the name of “a student who has a summer birthday” and “someone whose birth month starts with the letter J.” Through repeated exposure to probability in meaningful contexts, children can gradually develop their ideas about randomness and chance. They can see that, instead of “playing tricks on us,” probability does make sense.

Bibliography


Figure 3

Comparing spinners
Mia says she wants to choose the spinner that will give her the best chances of landing on stripes. What would you tell her?

Spinner A

Spinner B


